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ANTI-INVARIANT SUBMANIFOLDS WITH FLAT NORMAL CONNECTION

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1. Introduction

Anti-invariant, i.e., totally real, submanifolds of a Kaehlerian manifold have been studied by Blair [1], Chen [2], Houh [3], Kon [4], [10], [11], Ludden [5], [6], Ogiue [2], Okumura [5], [6], Yano [5], [6], [8], [9], [10], [11] and others. In particular, anti-invariant submanifolds of complex space forms have been recently studied by two of the present authors [10], [11].

The main purpose of the present paper is to study anti-invariant submanifolds of complex space forms with parallel mean curvature vector and flat normal connection, and to prove Theorems 1, 2, 3 and 4.

§ 2 contains preliminaries on field of frames convenient for the study of anti-invariant submanifolds of a complex space form. In § 3 we study anti-invariant submanifolds of a complex space form with flat normal connection, and prove some lemmas. The purpose of § 4 is to prove some theorems on anti-invariant submanifolds with parallel mean curvature vector and flat normal connection. In § 5, the last section, we give some examples of anti-invariant submanifold with parallel mean curvature vector and flat normal connection immersed in a complex projective n-space CP^n or complex n-space C^n, and prove our Theorems 3 and 4.

2. Preliminaries

Let $\overline{M}$ be a Kaehlerian manifold of complex dimension $n + p$ with almost complex structure J. A real n-dimensional Riemannian manifold M isometrically immersed in $\overline{M}$ is said to be *anti-invariant* or *totally real* in $\overline{M}$ if $JT_x(M) \subset T_x(M)^\perp$ for each point x of M, where $T_x(M)$ and $T_x(M)^\perp$ denote the tangent space and the normal space to M at x respectively.

We choose a local field of orthonormal frames $e_1, \cdots, e_n; e_{n+1}, \cdots, e_{n+p}$; $e_{1*} = Je_1, \cdots, e_{n*} = Je_n; e_{(n+1)*} = Je_{n+1}, \cdots, e_{(n+p)*} = Je_{n+p}$ in $\overline{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to this field of frames of $\overline{M}$, let $\omega^1, \cdots, \omega^n; \omega^{n+1}, \cdots, \omega^{n+p}; \omega^{1*}, \cdots, \omega^{n*}; \omega^{(n+1)*}, \cdots, \omega^{(n+p)*}$ be the field of dual frames. Unless otherwise stated, we use the following ranges of indices:

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$$A, B, C, D = 1, \cdots, n+p, 1^*, \cdots, (n+p)^* ,$$
$$i, j, k, l, t, s = 1, \cdots, n ,$$
$$a, b, c, d = n+1, \cdots, n+p, 1^*, \cdots, (n+p)^* ,$$
$$\alpha, \beta, \gamma = n+1, \cdots, n+p ,$$
$$\lambda, \mu, \nu = n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^* ,$$

and the convention that when an index appears twice in any term as a subscript and a superscript, it is understood that this index is summed over its range. Then the structure equations of $\overline{M}$ are given by

$$\text{(2.1)} \qquad \begin{aligned} &d\omega^A = -\omega^A_B \wedge \omega^B , \qquad \omega^A_B + \omega^B_A = 0 , \\ &\omega^i_j + \omega^j_i = 0 , \quad \omega^i_j = \omega^{i*}_{j*} , \quad \omega^{i*}_j = \omega^{j*}_i , \end{aligned}$$

$$\text{(2.2)} \qquad \begin{aligned} &\omega^i_\alpha + \omega^\alpha_i = 0, \quad \omega^i_\alpha = \omega^{i*}_{\alpha*} , \quad \omega^{i*}_\alpha = \omega^{\alpha*}_i , \\ &\omega^\alpha_\beta + \omega^\beta_\alpha = 0 , \quad \omega^\alpha_\beta = \omega^{\alpha*}_{\beta*} , \quad \omega^{\alpha*}_\beta = \omega^{\beta*}_\alpha , \end{aligned}$$

$$\text{(2.3)} \qquad d\omega^A_B = -\omega^A_C \wedge \omega^C_B + \Phi^A_B , \qquad \Phi^A_B = \tfrac{1}{2}K^A_{BCD}\omega^C \wedge \omega^D .$$

When we restrict these forms to M, we have

$$\text{(2.4)} \qquad \omega^a = 0 .$$

Since $0 = d\omega^a = -\omega^a_i \wedge \omega^i$, by Cartan's lemma we can write ω^a_i as

$$\text{(2.5)} \qquad \omega^a_i = h^a_{ij}\omega^j , \qquad h^a_{ij} = h^a_{ji} .$$

From these formulas we obtain the following structure equations of M:

$$\text{(2.6)} \quad d\omega^i = -\omega^i_j \wedge \omega^j , \quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j , \quad \Omega^i_j = \tfrac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l ,$$

$$\text{(2.7)} \qquad R^i_{jkl} = K^i_{jkl} + \sum_a (h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk}) ,$$

$$\text{(2.8)} \qquad d\omega^a_b = -\omega^a_c \wedge \omega^c_b + \Omega^a_b , \qquad \Omega^a_b = \tfrac{1}{2}R^a_{bkl}\omega^k \wedge \omega^l ,$$

$$\text{(2.9)} \qquad R^a_{bkl} = K^a_{bkl} + \sum_i (h^a_{ik}h^b_{il} - h^a_{il}h^b_{ik}) .$$

The forms (ω^i_j) define the Riemannian connection of M, and the forms (ω^a_b) the connection induced in the normal bundle of M. From (2.2) and (2.5) it follows that

$$\text{(2.10)} \qquad h^i_{jk} = h^j_{ik} = h^k_{ij} ,$$

where we have written h^i_{jk} in place of h^{i*}_{jk} to simplify the notation. The second fundamental form of M is represented by $h^a_{ij}\omega^i\omega^j e_a$, and is sometimes denoted by its components h^a_{ij}. If the second fundamental form is of the form $\delta_{ij}(\sum_k h^a_{kk}e_a)/n$, then M is said to be *totally umbilical*. If h^a_{ij} is of the form $h^a_{ij} = (\sum_k h^a_{kk})\delta_{ij}/n$, then M is said *to be umbilical with respect to* e_a. We call $(\sum_k h^a_{kk}e_a)/n$ the *mean curvature vector* of M, and M is said to be *minimal* if its mean curvature vector vanishes identically, i.e., $\sum_k h^a_{kk} = 0$ for all a. We define the covariant derivative h^a_{ijk} of h^a_{ij} by

$$h^a_{ijk}\omega^k = dh^a_{ij} - h^a_{il}\omega^l_j - h^a_{lj}\omega^l_i + h^b_{ij}\omega^a_b \,. \tag{2.11}$$

The Laplacian Δh^a_{ij} of h^a_{ij} is defined to be

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} \,, \tag{2.12}$$

where we have defined h^a_{ijkl} by

$$h^a_{ijkl}\omega^l = dh^a_{ijk} - h^a_{ljk}\omega^l_i - h^a_{ilk}\omega^l_j - h^a_{ijl}\omega^l_k + h^b_{ijk}\omega^a_b \,. \tag{2.13}$$

In the sequel we assume that the second fundamental form of M satisfies equations of Codazzi:

$$h^a_{ijk} - h^a_{ikj} = 0 \,. \tag{2.14}$$

Then, from (2.13), we have

$$h^a_{ijkl} - h^a_{ijlk} = h^a_{it}R^t_{jkl} + h^a_{tj}R^t_{ikl} - h^b_{ij}R^a_{bkl} \,. \tag{2.15}$$

On the other hand, (2.12) and (2.14) imply that

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} = \sum_k h^a_{kijk} \,. \tag{2.16}$$

From (2.14), (2.15) and (2.16) it follows that

$$\Delta h^a_{ij} = \sum_k (h^a_{kkij} + h^a_{kt}R^t_{ijk} + h^a_{ti}R^t_{kjk} - h^b_{ki}R^a_{bjk}) \,. \tag{2.17}$$

Therefore we have

$$\sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = \sum_{a,i,j,k} (h^a_{ij}h^a_{kkij} + h^a_{ij}h^a_{kt}R^t_{ijk} + h^a_{ij}h^a_{ti}R^t_{kjk} - h^a_{ij}h^b_{ki}R^a_{bjk}) \,. \tag{2.18}$$

If the ambient manifold $\overline{M}$ is of constant holomorphic sectional curvature c, then the Riemannian curvature tensor K^A_{BCD} of $\overline{M}$ is of the form

$$K^A_{BCD} = \tfrac{1}{4}c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) \,, \tag{2.19}$$

and the second fundamental form of M satisfies equations (2.14) of Codazzi.

3. Flat normal connection

In this section we study the normal connection of a real n-dimensional anti-invariant submanifold M of a complex space form $\bar{M}^{n+p}(c)$, that is, of a complex $(n+p)$-dimensional Kaehlerian manifold $\bar{M}$ of constant holomorphic sectional curvature c.

If $R^{a}_{bkl} = 0$ for all indices, then the normal connection of M is said to be *flat*.

From (2.19) we see, first of all, that

$$K^{\lambda}_{t^*kl} = 0 \,, \quad K^{\lambda}_{jkl} = 0 \,, \quad K^{\lambda}_{\mu kl} = 0 \,. \tag{3.1}$$

If the normal connection of M is flat, then (2.9) and (3.1) imply that

$$\sum_i (h^{\lambda}_{ik}h^{t}_{il} - h^{\lambda}_{il}h^{t}_{ik}) = 0 \,, \qquad \sum_i (h^{\lambda}_{ik}h^{\mu}_{il} - h^{\lambda}_{il}h^{\mu}_{ik}) = 0 \,. \tag{3.2}$$

Moreover, using (2.9) and (2.10), we see that

$$\sum_i (h^{t}_{ik}h^{s}_{il} - h^{t}_{il}h^{s}_{ik}) = \sum_i (h^{i}_{tk}h^{i}_{sl} - h^{i}_{tl}h^{i}_{sk}) = -\tfrac{1}{4}c(\delta_{tk}\delta_{sl} - \delta_{tl}\delta_{sk}) \,. \tag{3.3}$$

Proposition 1. *Let M be an n-dimensional $(n > 1)$ anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$. If the normal connection of M is flat, and M is umbilical with respect to some e_{t^*}, then $c = 0$.*

Proof. If M is umbilical with respect to e_{t^*}, then the second fundamental form h^{t}_{ij} is of the form $h^{t}_{ij} = (\sum_k h^{t}_{kk})\delta_{ij}/n$. Thus we have

$$\sum_i (h^{t}_{ik}h^{s}_{il} - h^{t}_{il}h^{s}_{ik}) = 0 \,.$$

From this and (3.3) we see that $c = 0$.

Lemma 1. *Let M be an n-dimensional anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$. If the normal connection of M is flat, then we have*

$$R^{i}_{jkl} = \sum_{\lambda} (h^{\lambda}_{ik}h^{\lambda}_{jl} - h^{\lambda}_{il}h^{\lambda}_{jk}) \,. \tag{3.4}$$

Proof. From (2.7) and (2.9) we find

$$\begin{aligned} R^{i}_{jkl} &= \tfrac{1}{4}c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_t (h^{i}_{tk}h^{j}_{tl} - h^{i}_{tl}h^{j}_{tk}) \\ &\quad + \sum_{\lambda} (h^{\lambda}_{ik}h^{\lambda}_{jl} - h^{\lambda}_{il}h^{\lambda}_{jk}) \\ &= R^{i^*}_{j^*kl} + \sum_{\lambda} (h^{\lambda}_{ik}h^{\lambda}_{jl} - h^{\lambda}_{il}h^{\lambda}_{jk}) \,. \end{aligned}$$

Since the normal connection of M is flat, we have $R^{i^*}_{j^*kl} = 0$ and hence (3.4).

In the sequel, we put $A_a = (h^{a}_{ij})$, A_a being a symmetric matrix.

Lemma 2. *Let M be an n-dimensional anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(c \neq 0)$. If the normal connection of M is flat, then M is umbilical with respect to all e_λ.*

Proof. From (3.2) we see that $A_\lambda A_\mu = A_\mu A_\lambda$ and $A_\lambda A_1 = A_1 A_\lambda$ for all λ and μ. Thus we can choose a local field of orthonormal frames with respect to which A_1 and all A_λ are diagonal, i.e.,

$$(3.5)\qquad A_1 = \begin{pmatrix} h^1_{11} & & 0 \\ & \ddots & \\ 0 & & h^1_{nn} \end{pmatrix}, \qquad A_\lambda = \begin{pmatrix} h^\lambda_{11} & & 0 \\ & \ddots & \\ 0 & & h^\lambda_{nn} \end{pmatrix}.$$

Putting $t = l$ and $k = 1$ in the first equation of (3.2) and using (3.5), we find

$$(3.6)\qquad (h^\lambda_{11} - h^\lambda_{tt})h^1_{tt} = 0 .$$

On the other hand, putting $t = k = 1$ and $s = l \neq 1$ in (3.3) and using (3.5), we have

$$(3.7)\qquad (h^1_{ll} - h^1_{11})h^1_{ll} = -\tfrac{1}{4}c .$$

Since $c \neq 0$, (3.7) implies that $h^1_{ll} \neq 0$. From this fact and (3.6) we see that $h^\lambda_{11} = h^\lambda_{tt}$ $(t = 2, \cdots, n)$ for all λ. Thus M is umbilical with respect to e_λ for all λ.

Lemma 3. *Let M be an n-dimensional anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(c \neq 0)$. If the normal connection of M is flat, then we have*

$$(3.8)\qquad R^i_{jkl} = \frac{1}{n^2}\sum_\lambda (\mathrm{Tr}\, A_\lambda)^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) .$$

Proof. From Lemma 2 we see that $h^\lambda_{ij} = (\mathrm{Tr}\, A_\lambda)\delta_{ij}/n$ for all λ. Therefore (3.4) implies (3.8).

If, in Lemma 3, $n \geq 3$, then $\sum_\lambda (\mathrm{Tr}\, A_\lambda)^2$ is constant. Therefore we have

Proposition 2. *Let M be an n-dimensional $(n \geq 3)$ anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(c \neq 0)$. If the normal connection of M is flat, then M is of constant curvature.*

If M is minimal, then $\mathrm{Tr}\, A_\lambda = 0$ for all λ. Thus we have, by (3.8),

Proposition 3. *Let M be an n-dimensional anti-invariant minimal submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(c \neq 0)$. If the normal connection of M is flat, then M is flat.*

4. Parallel mean curvature vector

Using the results obtained in the previous section, we can prove

Theorem 1. *Let M be an n-dimensional $(n \geq 3)$ anti-invariant submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(c \neq 0)$ with parallel mean curvature vector. If* *the normal connection of M is flat, then M is a flat anti-invariant submanifold of some $\bar{M}^n(c)$ in $\bar{M}^{n+p}(c)$, where $\bar{M}^n(c)$ is a totally geodesic complex submanifold of $\bar{M}^{n+p}(c)$ of complex dimension n.*

Proof. Since $n \geq 3$, $\sum_\lambda (\mathrm{Tr}\, A_\lambda)^2$ is constant. On the other hand, from (2.7) and (3.8), we have

$$\frac{n-1}{n} \sum_\lambda (\mathrm{Tr}\, A_\lambda)^2 = \frac{1}{4} n(n-1)c + \sum_a (\mathrm{Tr}\, A_a)^2 - \sum_{a,i,j} (h_{ij}^a)^2 . \tag{4.1}$$

Therefore the square of the length of the second fundamental form of M is constant, i.e., $\sum_{a,i,j} (h_{ij}^a)^2 =$ constant. From this we see that

$$\sum_{a,i,j,k} (h_{ijk}^a)^2 + \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = \tfrac{1}{2}\Delta \sum_{a,i,j} (h_{ij}^a)^2 = 0 . \tag{4.2}$$

Substituting (3.8) into (2.18) and using (4.2), we obtain

$$\begin{aligned} \sum_{a,i,j,k} (h_{ijk}^a)^2 &= -\frac{1}{n^2} \sum_\lambda (\mathrm{Tr}\, A_\lambda)^2 \sum_{a,i,j} [n(h_{ij}^a)^2 - h_{ii}^a h_{jj}^a] \\ &= -\frac{1}{n^2} \sum_\lambda (\mathrm{Tr}\, A_\lambda)^2 \sum_{t,i,j} [n(h_{ij}^t)^2 - h_{ii}^t h_{jj}^t] \\ &= -\frac{1}{n^2} \sum_\lambda (\mathrm{Tr}\, A_\lambda)^2 \sum_t \Big[\sum_{i>j} (h_{ii}^t - h_{jj}^t)^2 + n \sum_{i \neq j} (h_{ij}^t)^2 \Big] . \end{aligned} \tag{4.3}$$

To get the second line of (4.3), we have used Lemma 2. Since M is not umbilical with respect to each e_{t*} by Proposition 1 and $c \neq 0$ by the assumption, we have $\sum_{i>j} (h_{ii}^t - h_{jj}^t)^2 > 0$. Therefore we see that $h_{ijk}^a = 0$, that is, the second fundamental form of M is parallel and $\mathrm{Tr}\, A_\lambda = 0$, which implies that $A_\lambda = 0$ for all λ. From these and the fundamental theorem of submanifolds, M is an anti-invariant submanifold of $\bar{M}^n(c)$, where $\bar{M}^n(c)$ is a totally geodesic complex submanifold of $\bar{M}^{n+p}(c)$ of complex dimension n. Moreover, since $A_\lambda = 0$ for all λ, Lemma 3 shows that M is flat. From these considerations we have our assertion.

When $n = 2$, we need the assumption that M is compact. In this case we have

Theorem 2. *Let M be a compact anti-invariant surface of a complex space form $\bar{M}^{2+p}(c)$ $(c \neq 0)$ with parallel mean curvature vector. If the normal connection of M is flat, then M is a flat anti-invariant surface of some $\bar{M}^2(c)$ in $\bar{M}^{2+p}(c)$, where $\bar{M}^2(c)$ is a complex 2-dimensional totally geodesic submanifold of $\bar{M}^{2+p}(c)$.*

Proof. Since M is compact, we have

$$\int_M \sum_{a,i,j,k} (h_{ijk}^a)^2 {*1} = -\int_M \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a {*1} .$$

Using this and an argument quite similar to that used in the proof of Theorem

1, we have our assertion.

When $c = 0$, we have the following result under an additional assumption on A_λ.

Proposition 4. *Let M be an n-dimensional ($n \geq 3$) anti-invariant submanifold of a flat complex space form $\bar{M}^{n+p}(0)$ with parallel mean curvature vector and flat normal connection. If M is umbilical with respect to all e_λ, then either M is a flat anti-invariant submanifold of some $\bar{M}^n(0)$ in $\bar{M}^{n+p}(0)$, where $\bar{M}^n(0)$ is a flat totally geodesic complex submanifold of $\bar{M}^{n+p}(0)$, or M is a totally umbilical anti-invariant submanifold.*

Proof. From the assumption and (3.4) we have (3.8), so that (4.3) holds. If $\operatorname{Tr} A_\lambda = 0$ for all λ, then by (3.8) M is flat and immersed in some $\bar{M}^n(0)$ as an anti-invariant submanifold. If $\operatorname{Tr} A_\lambda \neq 0$ for some λ, then we have

$$\sum_t \left[\sum_{i>j} (h^t_{ii} - h^t_{jj})^2 + n \sum_{i \neq j} (h^t_{ij})^2\right] = 0 .$$

From this we conclude that $h^t_{ii} = h^t_{jj}$, $h^t_{ij} = 0$ $(i \neq j)$, so that each e_{t*} is an umbilical section. Thus M is totally umbilical.

Remark. If, in Proposition 4, M is totally umbilical and $n > 1$, then we have $A_t = 0$ for all t (see [10, p. 218]).

Proposition 5. *Let M be a compact anti-invariant surface of a flat complex space form $\bar{M}^{2+p}(0)$ with parallel mean curvature vector and flat normal connection. If M is umbilical with respect to all e_λ, then either M is a flat anti-invariant surface of some $\bar{M}^2(0)$ in $\bar{M}^{2+p}(0)$, where $\bar{M}^2(0)$ is a flat totally geodesic complex submanifold of $\bar{M}^{2+p}(0)$, or M is a totally umbilical anti-invariant submanifold.*

5. Flat anti-invariant submanifolds

In this section we give some examples of flat anti-invariant submanifolds with parallel mean curvature vector and flat normal connection immersed in CP^n or C^n.

First of all, we describe some properties of Riemannian fibre bundles.

Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors $(\phi, \xi, \eta, \bar{g})$ (cf. [7]). Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi , \quad \phi\xi = 0 , \quad \eta(\phi X) = 0 , \quad \eta(\xi) = 1 ,$$
$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y) , \quad \eta(X) = \bar{g}(X, \xi)$$

for any vector fields X and Y on $\bar{M}$. Moreover,

$$\bar{\nabla}_X \xi = \phi X , \qquad (\bar{\nabla}_X \phi) Y = -\bar{g}(X, Y)\xi + \eta(Y)X = \bar{R}(X, \xi)Y ,$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to $\bar{g}$, and $\bar{R}$ the Riemannian curvature tensor of $\bar{M}$. If M is regular, then there exists a fibering $\bar{\pi}: \bar{M} \to \bar{M}/\xi = \bar{N}$, $\bar{N}$ denoting the set of orbits of ξ, which is a real $2m$-dimensional Kaehlerian manifold. Let $(J, \bar{G})$ be the Kaehlerian structure of $\bar{N}$, and let $*$ denote the horizontal lift with respect to the connection η. Then we have

$$(5.1) \qquad (JX)^* = \phi X^* , \qquad \bar{g}(X^*, Y^*) = \bar{G}(X, Y)$$

for any vector fields X and Y on $\bar{N}$. Let $\bar{\nabla}'$ be the operator of covariant differentiation with respect to $\bar{G}$. Then

$$(5.2) \qquad (\bar{\nabla}'_X Y)^* = -\phi^2 \bar{\nabla}_{X^*} Y^* = \bar{\nabla}_{X^*} Y^* + \bar{g}(Y^*, \phi X^*)\xi .$$

Let M be an $(n+1)$-dimensional submanifold immersed in $\bar{M}$, and N an n-dimensional submanifold immersed in $\bar{N}$. In what follows we assume that M is tangent to the structure vector field ξ of $\bar{M}$, and there exists a fibration $\pi: M \to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & \bar{M} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ N & \xrightarrow{\ i'\ } & \bar{N} \end{array}$$

commutes, and the immersion i is a diffeomorphism on the fibres. Let g and G be the induced metric tensor fields of M and N respectively. Let ∇ (resp. ∇') be the operator of covariant differentiation with respect to g (resp. G). We denote by B (resp. B') the second fundamental form of the immersion i (resp. i') and the associated second fundamental forms of B and B' will be denoted by A and A' respectively. The Gauss formulas are written as

$$(5.3) \quad \bar{\nabla}'_X Y = \nabla'_X Y + B'(X, Y) , \qquad \bar{\nabla}_{X^*} Y^* = \nabla_{X^*} Y^* + B(X^*, Y^*) ,$$

for any vctor fields X and Y on N. From (5.2) and (5.3) we find that

$$(5.4) \qquad (\nabla'_X Y)^* = -\phi^2 \nabla_{X^*} Y^* , \qquad (B'(X, Y))^* = B(X^*, Y^*) .$$

Let D and D' be the operators of covariant differentiation with respect to the linear connections induced in the normal bundles of M and N respectively. For any tangent vector field X and any normal vector field V to N, we have the following Weingarten formulas

$$(5.5) \quad \bar{\nabla}'_X V = -A'_V X + D'_X V , \qquad \bar{\nabla}_{X^*} V^* = -A_{V^*} X^* + D_{X^*} V^* .$$

From (5.2) and (5.5) it follows that

$$(5.6) \qquad (A'_V X)^* = -\phi^2 A_{V^*} X^* , \qquad (D'_X V)^* = D_{X^*} V^* .$$

Since the structure vector field ξ of $\bar{M}$ is tangent to M, we have, for any vector field X tangent to M,

$$\bar{\nabla}_X \xi = \phi X = \nabla_X \xi + B(X, \xi) . \tag{5.7}$$

Putting $X = \xi$ in (5.7), we see that $B(\xi, \xi) = 0$. Now we take an orthonormal frame $e_1, \cdots, e_n$ for $T_{\pi(x)}(M)$. Then $e_1^*, \cdots, e_n^*, \xi$ form an orthonormal frame for $T_x(M)$. Let m and m' be the mean curvature vectors of M and N respectively. Then (5.4) and (5.9) imply

$$(m')^* = \sum_{i=1}^{n} (B'(e_i, e_i))^* = \sum_{i=1}^{n} B(e_i^*, e_i^*) + B(\xi, \xi) = m ,$$

that is,

$$(m')^* = m . \tag{5.8}$$

From (5.6) and (5.8) it follows that

$$(D'_X m')^* = D_{X^*} m . \tag{5.9}$$

In the sequel, we prove some lemmas for later use. First of all, we have, by (5.1),

Lemma 4. *M is an anti-invariant submanifold of $\bar{M}$ if and only if N is an anti-invariant submanifold of $\bar{N}$.*

Lemma 5. *Let M and N be anti-invariant submanifolds. Then the Riemannian curvature tensors R and R' of M and N respectively satisfy*

$$(R'(X, Y)Z)^* = R(X^*, Y^*)Z^* . \tag{5.10}$$

Proof. From (5.7) we see that the vector field ξ is parallel on M, i.e., $\nabla_X \xi = 0$ (see [12]). Thus we have

$$\eta(\nabla_{X^*} Y^*) = \nabla_{X^*} g(Y^*, \xi) - g(Y^*, \nabla_{X^*} \xi) = 0 .$$

From this and (5.4) we get $(\nabla'_X Y)^* = \nabla_{X^*} Y^*$, which implies

$$\begin{aligned}(R'(X, Y)Z)^* &= (\nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z - \nabla'_{[X,Y]} Z)^* \\ &= (\nabla_{X^*} \nabla_{Y^*} Z^* - \nabla_{Y^*} \nabla_{X^*} Z^* - \nabla_{[X^*,Y^*]} Z^*) \\ &= R(X^*, Y^*)Z^* .\end{aligned}$$

This gives (5.10).

From (5.10) and the fact that ξ is parallel on M, we have

Lemma 6. *Let M and N be anti-invariant submanifolds. Then M is flat if and only if N is flat.*

Lemma 7. *Let M be an $(n + 1)$-dimensional anti-invariant submanifold of a*

$(2n+1)$-dimensional Sasakian manifold $\bar{M}$, and N be an n-dimensional anti-invariant submanifold of a real $2n$-dimensional Kaehlerian manifold $\bar{N}$. Then the normal connection of M is flat if and only if the normal connection of N is flat.

Proof. From the assumption on the dimension we see that M is flat if and only if the normal connection of M is flat, and N is flat if and only if the normal connection of N is flat (cf. [10], [12]). From this and Lemma 6 we have our assertion.

Lemma 8. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$, and N be an n-dimensional anti-invariant submanifold of a real $2n$-dimensional Kaehlerian manifold $\bar{N}$. Then the mean curvature vector m of M is parallel if and only if the mean curvature vector m' of N is parallel.*

Proof. If m is parallel, (5.9) implies that m' is also parallel. Suppose that m' is parallel. Then, from (5.9), we have $D_{X^*}m = 0$. Therefore, we need only to prove that $D_\xi m = 0$.

First of all, by the Weingarten formula we have

$$D_X\phi Y = \bar{\nabla}_X\phi Y + A_{\phi Y}X = \eta(Y)X - g(X, Y)\xi + \phi\nabla_X Y + \phi B(X, Y) + A_{\phi Y}X \,.$$

Comparing the tangential and normal parts, we have

$$D_X\phi Y = \phi\nabla_X Y \,. \tag{5.11}$$

On the other hand, since $\bar{R}(X, \xi)Y = \eta(Y)X - g(X, Y)\xi$ is tangent to M for any tangent vector fields X, Y to M, we have

$$(\nabla_X B)(\xi, Y) = (\nabla_\xi B)(X, Y) \,. \tag{5.12}$$

We also have, from (5.7),

$$\nabla_X\xi = 0 \,, \qquad \phi X = B(X, \xi) \,. \tag{5.13}$$

Let $e_1, \cdots, e_{n+1}$ be an orthonormal frame for $T_x(M)$, and denote by the same letters local extension vector fields of this frame which are orthonormal and covariant constant with respect to ∇ at $x \in M$. Then, using (5.11), (5.12) and (5.13), we obtain

$$\begin{aligned} D_\xi m &= \sum_{i=1}^{n+1} (\nabla_\xi B)(e_i, e_i) = \sum_{i=1}^{n+1} (\nabla_{e_i} B)(\xi, e_i) \\ &= \sum_{i=1}^{n+1} D_{e_i}\phi e_i = \sum_{i=1}^{n+1} \phi\nabla_{e_i}e_i = 0 \end{aligned}$$

at each point x of M. Therefore we have $D_\xi m = 0$, and hence m is parallel.

Example 1. Let $S^1(r_i) = \{z_i \in C : |z_i|^2 = r_i^2\}$, $i = 1, \cdots, n+1$. We consider $M^{n+1} = S^1(r_1) \times \cdots \times S^1(r_{n+1})$ in C^{n+1} such that $r_1^2 + \cdots + r_{n+1}^2 = 1$. Then

M^{n+1} is a flat submanifold of S^{2n+1} with parallel mean curvature vector and flat normal connection. Moreover M is an anti-invariant submanifold of S^{2n+1} and tangent to the structure vector field ξ of S^{2n+1} (see [12]). Now we put $M^{n+1}/\xi = M^n_1$. Then the following diagram is commutative:

$$\begin{array}{ccc} M^{n+1} & \xrightarrow{\ i\ } & S^{2n+1} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ M^n_1 & \xrightarrow{\ i'\ } & CP^n . \end{array}$$

By Lemmas 4, 6, 7 and 8, M^n_1 is a flat anti-invariant submanifold of CP^n with parallel mean curvature vector and flat normal connection.

Example 2. Let $S^1(r_i) = \{z_i \in C : |z_i|^2 = r_i^2\}$, $i = 1, \cdots, n$. Then $M^n = S^1(r_1) \times \cdots \times S^1(r_n)$ is a flat anti-invariant submanifold of C^n (see [10]).

Theorem 3. *Let M be a compact n-dimensional anti-invariant submanifold of CP^{n+p} with parallel mean curvature vector. If the normal connection of M is flat, then M is M^n_1 of some CP^n in CP^{n+p}.*

Proof. By Theorems 1, 2, M is a flat anti-invariant submanifold of a CP^n in CP^{n+p}. Therefore, from Lemmas 4, 7, 8, $\pi^{-1}(M)$ is a flat anti-invariant submanifold of S^{2n+1} with parallel mean curvature vector and flat normal connection. By [12, Theorem 6.1] $\pi^{-1}(M)$ is $S^1(r_1) \times \cdots \times S^1(r_{n+1})$, $r_1^2 + \cdots + r_{n+1}^2 = 1$. Consequently M is congruent to M^n_1.

Theorem 4. *Let M be a compact n-dimensional anti-invariant submanifold of C^{n+p} with parallel mean curvature vector and flat normal connection. If M is umbilical with respect to all e_λ, then M is $S^1(r_1) \times \cdots \times S^1(r_n)$ in a C^n in C^{n+p} or $S^n(r)$.*

Proof. From Propositions 4, 5, we see that M is flat or totally umbilical. If M is flat, then, by a theorem of [10] and [11], M is $S^1(r_1) \times \cdots \times S^1(r_n)$ in a C^n in C^{n+p}. If M is totally umbilical, then M is obviously $S^n(r)$.

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